Michell J.H.の造波抵抗論文とその時代背景

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§1　船の波と造波抵抗

水面、つまり、水と空気の境界面を走る船の特徴の一つは美しい波紋を形成することである。飛行機から海面を走る船を見下ろすと、図1に示すように、船は八の字に広がる美しい波を曳きながら走るのが見える。速度が高くなればなるほど波は高くなるが、波の構成（波紋）は相似なように見える。また、船は一定の速度で走っているにもかかわらず、何故、次々と波が規則的に生まれてくるのだろうか。生じる波の高さと波長はどのように決まるのだろうか。詳細なメカニズムを見るために、船型試験水槽で長さ2m模型船を曳航した時の計測波紋図を図2に示す。船首波は山-谷-山-谷‥‥と規則的に生じ、波長は約1mで波頂を結ぶと略19°28′（理論値）を形成する。船尾波はやや乱れているが船首波と同じ傾向である。

図1 船の鳥瞰図
（https://jcolen.github.io/Files/RutgersF1

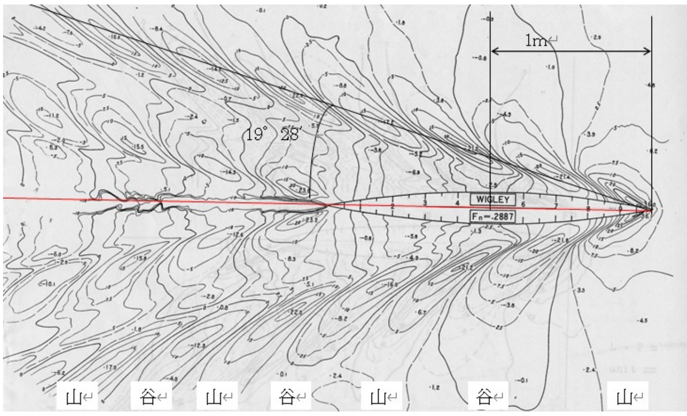

図2 Wigley模型船の計測波形（Fn=0.2887,V=1.278m/s）
（野澤和男：船_この巨大で力強い輸送システム（大阪大学出版会）
http://kansai-senior.sumomo.ne.jp/gallery/2021/211127-nozawa/index.html

船が走ると水抵抗を受ける。この水抵抗（全抵抗）は造波抵抗と粘性抵抗の和である。（ここで一時、粘性抵抗を忘れる）この美しい波形は実は、造波抵抗を作り出す原因でもある。
図3造波抵抗研究の方法で説明すると、
船（船体形状）①を模型試験水槽で所定の船速で曳航すると船に随伴する八の字型の船の波②が見られる。この波のエネルギーの全積分に相当する造波抵抗③に打ち勝つために実船ではエンジンがプロペラを回し造波抵抗（実際は全抵抗）に等しい推力を供給している。

❶造波抵抗直接計算

① 船体形状 → ②船の波 → ③造波抵抗値

❷船の起こす波形を研究し造波抵抗値の関係を研究、（❸実測波形を解析し造波抵抗を算定）

図3　造波抵抗研究の方法

❶造波抵抗直接計算：①船体形状を与えて直接、③造波抵抗値を計算　☜　Michellの研究
❷波形・造波抵抗計算：船（原因）と造波抵抗値（結果）の間に3次元的に広がる波を計測、波エネルギーの分布を研究　⇔　造波抵抗計算　☜　Havelockの研究
❸波形解析による造波抵抗算定：後続実測波形を計測・解析し実際に近い造波抵抗を算定　☜　波形解析による造波抵抗　←J.N.Newmanの研究(縦切り法)、後続研究として丸尾、池畑、野澤

§2　Michellの造波抵抗論文とその時代背景

J.H.Michell(1863‐1940)はイギリスからの移民の父を持つオーストラリアの数学者である。1898年に初めて船の造波抵抗に関する理論的計算式を導出した。（John.H.Michell：The wave resistance of a ship,Phil.Mag.(5)45(1898) 106-123）[1),2)]　日本の和号・歴史では、尊王攘夷論最盛期の文久3年（ペリー来航の10年後で明治維新の5年前）に生まれ、日独伊三国同盟締結の昭和15年まで生きた。造波抵抗計

算式は明治 31 年（日清・日露両戦争の間）に発表された。今生きていると 159 歳である。
流体運動を表す速度ポテンシャルφはラプラス方程式と線形自由表面条件（B/L が非常に小さい薄い船の近似）を満たすものとして、φの形を x,y,z を含む三角関数の積（分離型）で置き境界条件を満たすφをフーリエの方法で求め、さらに船体表面圧力を求めて前進方向成分を表面積分して造波抵抗を計算した。波の式には言及せず❶造波抵抗直接計算に着目している。後続の研究者が良く使うグリーン関数法が使われていないのは Michell よりやや先に生まれた George.Green(1793-1841)が考案したグリーン関数法が未だ定着していなかったのであろうか。

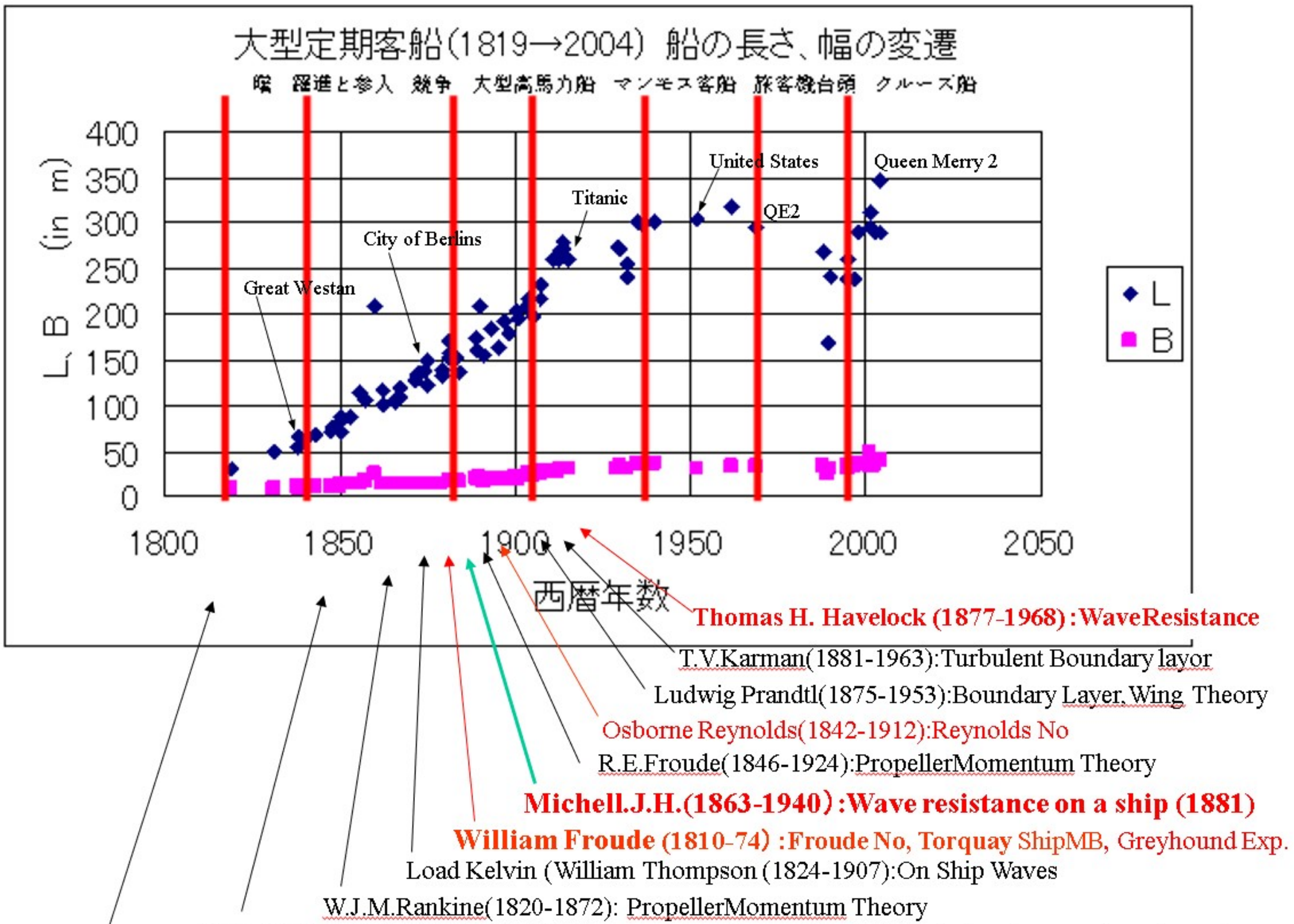

図4：大西洋航路定期客船：船の長さ、幅の変遷
（野澤和男著　船_この巨大で力強い輸送システム（阪大出版会）加筆修正）

商船発達史を見よう。1800 年代はイギリス産業革命の発展、拡大期でやがて大西洋航路定期客船のスピード競争が華々しく繰り広げられた時代にあたる。図 4 “大西洋航路定期客船：船の長さ,幅の変遷” [12)]にその発展を支えたであろう船舶流体力学上の著名な学者を列記した。その中に船体抵抗や造波抵抗の研究に名を遺す 4 人、William Froude, John. H. Michell、Osborne Reynolds, Thomas. H. Havelock がいる。

Froude は自邸内に船型試験水槽を設けて研究したことで有名で、実験船型学の立場から船の抵抗（剰余抵抗、摩擦抵抗）を研究した。造波抵抗の相似則で重要な Fn(フルード数)を見い出した。[3)] Michell は史上初の造波抵抗理論式を導出した。[1), 2)] Reynolds は粘性流の相似則に重要な Rn(レイノルズ数)を見い出し船や航空機の粘性抵抗研究の基礎を作った。[4)] 彼らの少し後の 1877 年生まれの Havelock は❷の立場から船の波と造波抵抗の関連を種々の角度から理論的に研究、発展させた。彼の膨大な論文は Collected Papers として ONR（USA）から出版され造波抵抗研究学徒の座右の書となった。[5)]

1970 年頃まで日本造船工業は建造量第 1 位として世界を牽引していた。各種商船の大型高馬力化、高速化を目指して流体力学研究者の造波抵抗研究 [6)、7)、8)] が進み、また造船各社は船型開発にしのぎを削った。船型研究者は競って上記の造波抵抗論文を勉強、研究し船舶設計に応用したがその基礎には Michell の理論があった。しかし薄船近似のため L/B が 10 以上の船にはよく適合するが L/B が 7 程度の実用船型には応用できなかった。多くの研究者は理論の改良や高次近似理論、波形解析法の応用 [9)] などを試みたが自由表面条件の強い非線形性や粘性影響のために実用の域に達しなかった。(興味ある論文も多くあるが

ここでは触れない）1985 年頃になるとコンピューターの著しい発展に伴い船舶への CFD(数値流体力学)の応用の機運が高まり主流になっていった。1995 年の産官学共同研究 SR222“大型肥大船の船尾流場推定法の高度化”[10] は船舶 CFD 元年と位置付けられる。

Michell 理論の薄い船の仮定がそのまま成り立つ“船”も出てきた。超高速多胴船である。多胴船は各船が薄船であっても復原性の問題が無く船として成立する。因みに、Austal の超高速三胴船“Benchijigua Express”の L/B は main hull が 12、side hull が 40 である。筆者らの研究によると、Michell 理論による造波抵抗係数 Cw は波形解析によるものと非常によく一致し Michell 理論が船型設計に十分有効であることが分かった。[11]

古色蒼然として孤高で美しく纏められた“Michell 造波抵抗式”は筆者にとって長年にわたり、研究、設計に活用した理論であり思い出が深い。最近、偶々Michell の論文をレビューしたがこの貴重な論文がいつまでも散逸せずに役立つように小文を付けて Upload することにした。 ⚓

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紹介 www：http://kansai-senior.sumomo.ne.jp/gallery/2021/211127-nozawa/index.html

§ 3 The Wave Resistance of a Ship by John.H.Michell

出典：：John.H.Michell :The wave resistance of a ship,Phil.Mag. (5)45 (1898) 106-123)

The formula was tested by drawing a system of continuous isothermals giving pv plotted against $v^{-\frac{1}{3}}$, and then the experimental values were put in as dots; there is on the whole a fair agreement between calculation and experiment, as may be seen on inspection of the diagram. It is generally difficult in these investigations to know how much may be reasonably allowed for experimental errors. Fortunately in this instance we have a clue to guide us, as Messrs. Ramsay and Young in testing their linear law published tables comparing pressures found with pressures calculated (*loc. cit.* pp. 438–440 and pp. 442–445), and from these it is seen that they were willing to allow over 1 per cent. as a possible experimental error. In this connexion they remark: "It is to be noticed that the greatest divergence is at the temperatures 250° and 280°, but the deviations are in opposite directions and must therefore be ascribed to experimental error" (*loc. cit.* p. 444).

I likewise found in testing my formula that the greatest divergence is at temperatures 280°·35 C. and 250° C., and that the deviations are in opposite directions, and therefore consider it justifiable to attribute them mainly to the same cause. For the remaining temperatures discrepancies occur fairly often of over 1 per cent., but none so great as 2 per cent., so that they still seem to lie within the limits of experimental error.

Finally we may infer that both the general conclusions obtained in the former paper with regard to isopentane hold good also in the case of ether.

XI. *The Wave-Resistance of a Ship.* *By* J. H. Michell.*

THE object of this paper is to give a general solution of the problem of the waves produced by a ship of given form moving with uniform velocity in an inviscid liquid, and to determine the consequent wave-resistance to the motion of the ship. The only assumption made as to the form of the ship is that the inclination of the tangent plane at any point of its surface to the vertical median plane is small. This condition is not satisfied near the bottom of the middle body of a modern ship, but it seems probable that this will not much affect either the waves produced or the resistance, for the waves arise rather from the parts at the bow and stern at which the tangent to the surface is inclined to the direction of the ship's motion, than from the approximately cylindrical

* Communicated by the Author.

middle body. The neglect of friction is probably of little consequence. The eddying water close to the side will no doubt slightly alter the virtual shape of the ship, but the change in the inclination of the virtual tangent plane, on which the wave-making depends, will, almost certainly, be very small. Further, the effect of viscosity in destroying the waves produced by the bow will modify to some extent the interaction of bow and stern waves; but, seeing that it is the waves of length comparable with that of the ship which chiefly give rise to the resistance, the effect must be small. The conclusion is, therefore, that the course followed by W. Froude, of considering frictional resistance and wave-resistance separately and adding the two, will probably give a close approximation to the truth.

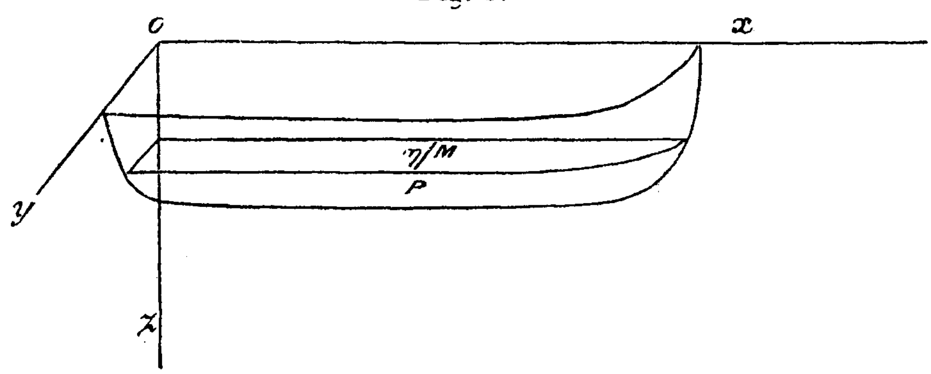
A summary of the experimental work on this question, as well as a sketch of the theoretical work of Russell, Rankine, and the two Froudes, is given in White's interesting 'Manual of Naval Architecture' (1894), chap. xi. Recent mathematics on the subject has been devoted chiefly to explaining, in a general way, the interesting wave-patterns observed, but exception must be made of the papers of Sir W. Thomson (Lord Kelvin), Phil. Mag. (1886–7), in which the critical speed of a canal boat, examined experimentally long before by Russell, was mathematically discussed. Reference may be made to Lamb's 'Hydrodynamics' (1895), chap. ix., and to Sir W. Thomson's 'Popular Lectures on Navigation' (1891), p. 450, for the discussion of wave-patterns.

None of these writers has, however, considered the waves produced by an actual ship, and the present paper is intended to supply the necessary investigation. The formula obtained for the wave-disturbance is rather complex, but that for the resistance is much simpler, as the most complicated term in the former represents a local disturbance not leading to any resistance in an inviscid liquid. There will therefore be no difficulty to those who have the necessary apparatus in making comparisons with experiment. As to general results, for deep water the theory leads to a resistance which increases with the velocity, in an oscillating manner, to a maximum and then decreases to zero as the velocity is indefinitely increased. That the resistance is an oscillating function of the velocity has been experimentally found by Mr. W. Froude and his son *, to whom also we owe the general explanation in terms of the interference of bow and stern waves. But the

* "On the Leading Phenomena of the Wave-making Resistance of Ships," Trans. Inst. Naval Architects, 1881.

ultimate vanishing of the resistance has not, so far as I know, been anticipated. From general considerations it is clear that, so far as the wave-form is concerned, the effect of increasing the velocity is the same as that of decreasing the acceleration of gravity, and, if gravity vanishes, there is no propagation of waves; but this is not quite the theorem to be obtained.

Fig. 1.



Take the vertical median plane of the ship as $y=0$, and the surface of the undisturbed water as $z=0$, the axis Ox being in the direction of motion of the ship and Oz vertically downwards. We may suppose the ship at rest and the water moving backwards with uniform velocity v apart from the wave-disturbance. The motion is assumed steady and the velocity potential written $-vx+\phi$. Since the inclination of the ship's surface to the plane $y=0$ is everywhere small, ϕ will be small, and we shall neglect the squares of the velocities due to ϕ in comparison with their first powers. At the surface of the water let ζ be the depression at (x, y) below the mean level. Then

$$\frac{d\phi}{dz}=-v\frac{d\zeta}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is the kinematic surface condition, and

$$p/\rho+\tfrac{1}{2}q^2-g\zeta=\text{const.}$$

the equation of pressure, which, since

$$q^2=\left(-v+\frac{d\phi}{dx}\right)^2+\left(\frac{d\phi}{dy}\right)^2+\left(\frac{d\phi}{dz}\right)^2$$

$$=v^2-2v\frac{d\phi}{dx}, \quad \text{(q. p.)}$$

gives

$$v\frac{d\phi}{dx}+g\zeta=0$$

and, therefore, with (1)

$$\frac{d\phi}{dz}=\frac{v^2}{g}\frac{d^2\phi}{dx^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

On account of the symmetry of the ship with respect to the median plane $y=0$, we have $d\phi/dy=0$ when $y=0$, except over the ship, where, if η is the semi-breadth at (x, z),

$$\frac{d\phi}{dy}=-v\frac{d\eta}{dx}$$

$$=-vf(x,z) \quad \text{(say)}, \quad . \quad . \quad (3)$$

and this condition is taken to hold at the plane $y=0$, instead of at the surface of the ship, the justification being the same as that for equation (1). Finally, $d\phi/dz=0$ at the bottom, $z=h$, of the water.

We now consider the solution for ϕ, in the part of the water where y is positive, with the given boundary conditions at $z=0$, $z=h$, $y=0$.

The typical term in the solution is

$$a\cos n(z-h)\cos(mx+\alpha)\cos(py+\beta),$$

where $m^2+n^2+p^2=0$. Here m must be taken real as the water extends from $x=-\infty$ to $x=+\infty$; n and p may be either real or imaginary, but if p is imaginary $[=ip']$ the last factor must take the form $e^{-p'y}$.

This term satisfies $d\phi/dz=0$ at $z=h$, and it also satisfies equation (2), if

$$n\tan nh=-v^2m^2/g. \quad . \quad . \quad . \quad . \quad (4)$$

This equation has an infinite number of real roots and *one* pure imaginary root given by

$$n'\tanh n'h=v^2m^2/g, \quad [n=in'].$$

We shall see that the imaginary root is alone responsible for the wave-making resistance. As for p it is always imaginary for the real roots of n, and is so for the imaginary root if $m>n'$.

The condition (3) will now require the expansion of the given function $f(x, z)$ in the form

$$\Sigma\Sigma a_{mn}\cos n(z-h)\cos(mx+\alpha),$$

where the summation with respect to m will take the form of an integral.

Suppose at first the function periodic in x so that

$$f(x+l, z)=f(x-l, z),$$

and put

$$f(x,z)=\Sigma_r\Sigma_n\left\{A_{rn}\cos\frac{\pi rx}{l}+B_{rn}\sin\frac{\pi rx}{l}\right\}\cos n(z-h),$$

where r is a positive integer.

By Fourier's method

$$\int_{-l}^{l}f(x,z)\cos\frac{\pi rx}{l}dx=l\,\Sigma_n A_{rn}\cos n(z-h),$$

$$\int_{-l}^{l}f(x,z)\sin\frac{\pi rx}{l}dx=l\,\Sigma_n B_{rn}\cos n(z-h),$$

where A_{on} is to be halved as usual.

Since the functions $\cos n(z-h)$ are all conjugate, as is easily proved, from these we get

$$\int_0^h\int_{-l}^{l}f(x,z)\cos\frac{\pi rx}{l}\cos n(z-h)\,dx\,dz=l\,A_{rn}\int_0^h\cos^2 n(z-h)dz,$$

$$=l\,A_{rn}\frac{1}{4n}(2nh+\sin 2nh)$$

and

$$\int_0^h\int_{-l}^{l}f(x,z)\sin\frac{\pi rx}{l}\cos n(z-h)\,dx\,dz=l\,B_{rn}\frac{1}{4n}(2nh+\sin 2nh);$$

where A_{oo} is to be halved; and the coefficients of the terms given by the imaginary roots, here, as always below, are got by putting $n=in'$; so that

$$\int_0^h\int_{-l}^{l}f(x,z)\cos\frac{\pi rx}{l}\cosh n(z-h)\,dx\,dz=l\,A_{rn'}\frac{1}{4n'}(2n'h+\sinh 2n'h)$$

and so for $B_{rn'}$.

Hence the theorem

$$f(x,z)=\Sigma_r\Sigma_n\frac{4n\cos n(z-h)}{l(2nh+\sin 2nh)}\int_{-l}^{l}\int_0^h f(\xi,\zeta)\cos\frac{\pi r}{l}(\xi-x)\times\cos n(\zeta-h)d\zeta\,d\xi$$

$$+\Sigma_r\frac{4n'\cosh n'(z-h)}{l(2n'h+\sinh 2n'h)}\int_{-l}^{l}\int_0^h f(\xi,\zeta)\cos\frac{\pi r}{l}(\xi-x)\times\cosh n(\zeta-h)d\zeta\,d\xi.$$

Now let l become infinite, and putting

$$\pi r/l=m$$

$$\pi/l=dm,$$

we get

$$f(x,z) = \frac{4}{\pi}\Sigma_n \int_0^{\infty}\int_{-\infty}^{\infty}\int_0^h f(\xi,\zeta)\, \frac{n\cos n(z-h)\cos n(\zeta-h)}{2nh+\sin 2nh} \times \cos m(\xi-x)\,d\zeta\, d\xi\, dm$$

$$+ \frac{4}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty}\int_0^h f(\xi,\zeta)\, \frac{n'\cosh n'(z-h)\cosh n'(\zeta-h)}{2n'h+\sinh 2n'h} \times \cos m(\xi-x)\,d\zeta\, d\xi\, dm.$$

In particular, suppose the depth of water infinite, we have then

$$nh = r\pi + \epsilon,$$

$$hdn = \pi,$$

$$\tan nh = \tan \epsilon,$$

$$\cos n(z-h) = (-)^r \cos(nz-\epsilon),$$

$$2n + (\sin 2nh)/h = 2n,$$

$$n\tan\epsilon = -km^2 \quad [k = v^2/g],$$

$$\sin 2\epsilon = \frac{-2km^2 n}{n^2+k^2m^4},$$

$$\cos 2\epsilon = \frac{n^2-k^2m^4}{n^2+k^2m^4}.$$

Also

$$\tanh n'h = 1,$$

$$n = km^2,$$

$$\frac{\cosh n'(z-h)\cosh n'(\zeta-h)}{2n'h+\sinh 2n'h} = \tfrac{1}{2}e^{-n'(z+\zeta)}.$$

The result receives some confirmation from what has been observed with torpedo-boats at high speeds. It has been found that the *total* resistance varies as a power of the velocity which at first is nearly the second, but which, increasing to a maximum, ultimately becomes less than the second *. A very simple investigation, given below, shows that in shallow water, if we neglect all but the long waves, the wave-resistance varies ultimately as the first power of the velocity.

I may mention that somewhat similar work to that of the present paper gives a theory of the damping of the oscillations of ships due to wave-making. This I hope to give in a subsequent paper.

* White, p. 470.

Substituting, we get

$$f(x, z) = \frac{2}{\pi^2}\int_0^\infty\int_0^\infty\int_0^\infty\int_{-\infty}^\infty f(\xi, \zeta)\cos(nz-\epsilon)\cos(n\zeta-\epsilon) \times \cos m(\xi-x)d\xi\, d\zeta\, dm\, dn$$

$$+ \frac{2v^2}{\pi g}\int_0^\infty\int_0^\infty\int_{-\infty}^\infty f(\xi, \zeta)m^2e^{-km^2(z+\zeta)}\cos m(\xi-x)d\xi\, d\zeta\, dm,$$

which is the theorem on which the rest of the present paper is built. It is curiously easy to give an *à posteriori* proof of the theorem. Using the value of ϵ given above we find

$$\cos(nz-\epsilon)\cos(n\zeta-\epsilon) = \cos nz\cos n\zeta - \cos n(z+\zeta)\frac{k^2m^4}{n^2+k^2m^4} - \sin n(z+\zeta)\frac{km^2n}{n^2+k^2m^4}.$$

Integrate the last two terms with respect to n, viz.

$$k^2m^4\int_0^\infty \frac{\cos n(z+\zeta)}{n^2+k^2m^4}dn = \frac{\pi}{2}km^2e^{-km^2(z+\zeta)},$$

$$km^2\int_0^\infty \frac{n\sin n(z+\zeta)}{n^2+k^2m^4}dn = \frac{\pi}{2}km^2e^{-km^2(z+\zeta)},$$

and the quadruple integral becomes

$$\frac{2}{\pi^2}\int_0^\infty\int_0^\infty\int_0^\infty\int_{-\infty}^\infty f(\xi, \zeta)\cos nz\cos n\zeta\cos m(\xi-x)d\xi\, d\zeta\, dm\, dn$$

$$-\frac{2k}{\pi}\int_0^\infty\int_0^\infty\int_{-\infty}^\infty f(\xi, \zeta)m^2e^{-km^2(z+\zeta)}\cos m(\xi-x)d\xi\, d\zeta\, dm.$$

The former integral is $f(x, z)$, and the latter disappears with the triple integral in the given formula.

Considering now, for simplicity, the water infinitely deep, it appears at once that the required solution for ϕ is

$$\phi = \frac{2v}{\pi^2}\int_0^\infty\int_0^\infty\int_0^\infty\int_{-\infty}^\infty f(\xi, \zeta)\frac{\cos(nz-\epsilon)\cos(n\zeta-\epsilon)}{\sqrt{m^2+n^2}}\cos m(\xi-x)e^{-\sqrt{m^2+n^2}y} \times d\xi\, d\zeta\, dm\, dn$$

$$-\frac{2v^3}{\pi g}\int_{g/v^2}^\infty\int_0^\infty\int_{-\infty}^\infty f(\xi, \zeta)\frac{me^{-m^2v^2(z+\zeta)/g}}{\sqrt{m^2v^4/g^2-1}}\sin\{m(x-\xi)+m\sqrt{m^2v^4/g^2-1}\,y\} \times d\xi\, d\zeta\, dm$$

$$+\frac{2v^3}{\pi g}\int_0^{g/v^2}\int_0^\infty\int_{-\infty}^\infty f(\xi, \zeta)\frac{me^{-m^2v^2(z+\zeta)/g}}{\sqrt{1-m^2v^4/g^2}}\cos m(\xi-x)e^{-m\sqrt{1-m^2v^4/g^2}\,y}\, d\xi\, d\zeta\, dm; \quad \ldots\ldots (5)$$

for this gives

$$\frac{d\phi}{dy} = -vf(x,z) = -v\frac{d\eta}{dx} \text{ when } y=0.$$

In this expression attention must be called to the factor

$$\sin\{m(x-\xi) + m\sqrt{m^2v^4/g^2-1}\,y\}$$

in the second integral. This form is not required to satisfy the boundary conditions formulated above; and it is evident that the solution is to a certain extent indeterminate with those conditions, for we may superpose any system of free waves symmetrical with respect to $y=0$ on a particular solution satisfying them. The form of the factor in question is chosen in order to make the elementary diverging waves trail aft; in other words, to satisfy the condition that the ship advances into still water.

Leaving the reduction of the integrals on one side, for the present, we proceed to calculate the wave-resistance (R).

Let δp be the increase of pressure due to the wave-disturbance. Then

$$R = -2\iint \delta p \frac{d\eta}{dx} dx\, dz,$$

the double integral extending over the median plane of the ship. Now measuring from the undisturbed surface

$$p = \pi + g\rho z - \tfrac{1}{2}\rho q^2 + \tfrac{1}{2}\rho v^2,$$

and therefore $$\delta p = \rho v \frac{d\phi}{dx}, \qquad \text{(q.p.)}$$

so that $$R = -2\rho v \iint \frac{d\phi}{dx}\frac{d\eta}{dx} dx\, dz.$$

Substituting the value of $d\phi/dx$, we see that the first and third integrals in the expression for ϕ add nothing to the resistance because

$$\iiiint f(\xi,\zeta) f(x,z) \sin m(x-\xi)\, d\xi\, d\zeta\, dx\, dz = 0,$$

on account of the odd factor $\sin m(x-\xi)$, and hence

$$R = \frac{4\rho v^4}{\pi g}\int_{g/v^2}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} f(x,z) f(\xi,\zeta) \frac{m^2 e^{-m^2v^2(z+\zeta)/g}}{\sqrt{m^2v^4/g^2-1}} \cos m(x-\xi)\, dx\, dz\, d\xi\, d\zeta\, dm$$

$$= \frac{4\rho v^4}{\pi g}\int_{g/v^2}^{\infty} (I^2+J^2) \frac{m^2 dm}{\sqrt{m^2v^4/g^2-1}}$$

$$= \frac{4\rho g^2}{\pi v^2}\int_1^{\infty} (I^2+J^2) \frac{\lambda^2 d\lambda}{\sqrt{\lambda^2-1}}, \qquad \qquad (6)$$

where
$$\lambda = mv^2/g,$$
and

$$I = \int_0^\infty \int_{-\infty}^\infty f(x, z) e^{-\lambda^2 gz/v^2} \cos \lambda gx/v^2 \, dx \, dz,$$

$$J = \int_0^\infty \int_{-\infty}^\infty f(x, z) e^{-\lambda^2 gz/v^2} \sin \lambda gx/v^2 \, dx \, dz.$$

If the ship is similarly formed at bow and stern $I=0$, the origin being at midship.

We can now prove that the resistance vanishes when the velocity is infinite.

Observe that

$$\int_0^\infty f(x, z) e^{-\lambda^2 gz/v^2} dz = F(x) \int_0^\infty e^{-\lambda^2 gz/v^2} dz$$
$$= \frac{1}{\lambda^2} \frac{v^2}{g} F(x), \quad . \quad . \quad . \quad . \quad (7)$$

where $F(x)$ is less than the greatest value of $f(x, z)$ for a given value of x; and, therefore, if we substitute a large number t instead of ∞ as the upper limit of λ, the part of R neglected is of order not greater than

$$v^2 \int_t^\infty \frac{d\lambda}{\lambda^3} \quad \text{or} \quad v^2 t^{-2},$$

and this vanishes when $v=\infty$ if we take

$$t = \left(\frac{\quad}{g} \right)^{2/3}.$$

In the part of R retained $\lambda g/v^2$ is small throughout, so that we may expand the circular functions and write

$$I = \int_0^\infty \int_{-\infty}^\infty f(x, z) e^{-\lambda^2 gz/v^2} dx \, dz$$
$$- \frac{1}{2} \lambda^2 \frac{g^2}{v^4} \int_0^\infty \int_{-\infty}^\infty f(x, z) x^2 e^{-\lambda^2 gz/v^2} dx \, dz + \ldots$$

and

$$J = \lambda \frac{g}{v^2} \int_0^\infty \int_{-\infty}^\infty f(x, z) x e^{-\lambda^2 gz/v^2} dx \, dz$$
$$- \frac{1}{6} \lambda^3 \frac{g^3}{v^6} \int_0^\infty \int_{-\infty}^\infty f(x, z) x^3 e^{-\lambda^2 gz/v^2} dx \, dz + \ldots$$

Now

$$\int_{-\infty}^{\infty} f(x, z)dx = \int_{-\infty}^{\infty} \frac{d\eta}{dx}\,dx = 0,$$

the ship being of finite length and $\eta=0$ at both ends. Hence, using the formula (7)

$$I = A\frac{g}{v^2} + B\lambda^2\left(\frac{g}{v^2}\right)^3 + \ldots .$$

$$J = A'\frac{1}{\lambda} + B'\lambda\left(\frac{g}{v^2}\right)^2 + \ldots .$$

and

$$R = \frac{4\rho g^2}{\pi v^2}\int_1^t \left[A''\frac{1}{\lambda^2} + B''\left(\frac{g}{v^2}\right)^2 + C''\lambda^2\left(\frac{g}{v^2}\right)^4 + \ldots\right]\frac{\lambda^2 d\lambda}{\sqrt{\lambda^2-1}}.$$

The successive terms are of orders

$$\frac{1}{v^2}\log v^2, \qquad v^{-10/3}, \quad ..$$

and all vanish when $v=\infty$. The resistance therefore ultimately vanishes. Of course this result is only proved for a ship which is very short in comparison with the depth of the water.

We now proceed to the reduction of the integral which gives the resistance due to two elements of the surface.

Consider two elementary areas σ, σ' at (x, z), (x', z') on the side of the ship, and let θ, θ' be the inclinations of the horizontal lines in these areas to the axis of x. The resistance due to these two elements is

$$\frac{8\rho g^2}{\pi v^2}\sigma\,\sigma'\,\theta\,\theta'\int_1^\infty e^{-\lambda^2 g(z+z')/v^2}\cos\lambda g(x-x')/v^2\frac{\lambda^2}{\sqrt{\lambda^2-1}}\,d\lambda,$$

or, say,

$$\frac{8\rho g^2}{\pi v^2}\sigma\,\sigma'\,\theta\,\theta'\int_1^\infty e^{\quad 2}\cos \varepsilon\lambda\frac{\lambda^2}{\sqrt{\lambda^2-1}}d\lambda.$$

Now, writing for the moment $z\equiv x+iy$, and taking the integral

$$\int\frac{e^{isz}}{\sqrt{1-z^2}}dz$$

around the circuit enclosing the region x and y positive, we get

$$\int^{\infty}\frac{e}{\sqrt{1-x^2}}dx + i\int_1^\infty\frac{e^{isx}}{\sqrt{x^2-1}}dx - i\int_0^\infty\frac{e^{-sx}}{\sqrt{1+x^2}}dx = 0,$$

and, therefore *, realizing,

$$\int_1^\infty \frac{\cos s\lambda}{\sqrt{\lambda^2-1}}\,d\lambda=\int_0^\infty \frac{e^{-s\lambda}}{\sqrt{1+\lambda^2}}d\lambda-\int_0^1\frac{\sin s\lambda}{\sqrt{1-\lambda^2}}\,d\lambda$$

$$=\int_0^{\infty} e^{-s\sinh\theta}\,d\theta-\int_0^{\frac{\pi}{2}}\sin(s\sin\theta)\,d\theta$$

$$=\kappa J_0(s)-Y_0(s), \quad . \quad . \quad . \quad . \quad (8)$$

where $\kappa=\log 2-\gamma=\cdot 11593\ldots.$
From which

$$-\frac{d^2}{ds^2}\int_1^\infty \frac{\cos s\lambda}{\sqrt{\lambda^2-1}}=-\{\kappa J_0''(s)-Y_0''(s)\}$$

$$=\{\kappa J_0(s)-Y_0(s)\}$$

$$-\frac{1}{s}\{\kappa J_1(s)-Y_1(s)\},$$

since

$$J_0''(s)+J_0'(s)/s+J_0(s)=0$$

$$J_0'(s)=-J_1(s),$$

and so for $Y_0(s)$.

Now taking

$$H=\int_1^\infty e^{-r\lambda^2}\cos s\lambda\frac{d\lambda}{\sqrt{\lambda^2-1}},$$

and putting

$$e^{-r\lambda^2}=\frac{1}{\sqrt{\pi r}}\int_0^\infty e^{-\mu^2/4r}\cos\lambda\mu\,d\mu,$$

$$H=\frac{1}{2\sqrt{\pi r}}\int_0^\infty\int_1^\infty e^{-\mu^2/4r}\{\cos\lambda(s+\mu)+\cos\lambda(s-\mu)\}\frac{d\lambda d\mu}{\sqrt{\lambda^2-1}}$$

$$=\frac{1}{2\sqrt{\pi r}}\int_0^\infty e^{-\mu^2/4r}\{\kappa J_0(s+\mu)-Y_0(s+\mu)$$

$$+\kappa J_0(s-\mu)-Y_0(s-\mu)\}d\mu, \quad . \quad . \quad (9)$$

from equation (8).

From which H can be readily calculated by mechanical quadrature in the case in which $4r$ is not large, and this is the case for ordinary ships. Elaborate tables of J_0 and J_1

* See Gray and Mathews, 'Bessel Functions,' p. 65, and Ex. 18, p. 230. The formula (8) was given by Weber.

are now available, and tables* of Y_0 and Y_1 have been calculated by Mr. B. A. Smith, who has kindly prepared tables of $\kappa J_0 - Y_0$ and $\kappa J_1 - Y_1$, appended to the present paper.

We now have

$$-\frac{d^2H}{ds^2} = \frac{1}{2\sqrt{\pi r}} \int_0^\infty e^{-\mu^2/4r} \{F(s+\mu) + F(s-\mu)\} d\mu,$$

where

$$F(s+\mu) = \kappa J_0(s+\mu) - Y_0(s+\mu) - \frac{1}{s+\mu}\{\kappa J_1(s+\mu) - Y_1(s+\mu)\},$$

and the expression for the mutual resistance is

$$\frac{4\rho g^2}{\pi^{3/2} v^2 \sqrt{r}} \sigma \sigma' \theta \theta' \int_0^\infty e^{-\mu^2/4r} \{F(s+\mu) + F(s-\mu)\} d\mu,$$

where

$$r = g(z+z')/v^2$$
$$s = g(x-x')/v^2.$$

For elements at opposite ends of the ship s will in general be large compared with unity and with $\sqrt{4r}$, and in this case we can put

$$\kappa J_0(s+\mu) - Y_0(s+\mu) = \sqrt{\frac{\pi}{2(s+\mu)}} \sin\left\{\frac{\pi}{4} - (s+\mu)\right\} \text{ q. p.}$$

and so for $(s-\mu)$, and then approximately

$$H = \frac{1}{\sqrt{2rs}} \sin\left(\frac{\pi}{4} - s\right) \int_0^\infty e^{-\mu^2/4r} \cos \mu \, d\mu$$
$$= \sqrt{\frac{\pi}{2s}} \cdot \sin\left(\frac{\pi}{4} - s\right) e^{-r},$$

and the resistance is

$$\frac{8\rho g^2}{\pi v^2} \sigma \sigma' \theta \theta' \sqrt{\frac{\pi}{2s}} \cdot \sin\left(\frac{\pi}{4} - s\right) e^{-r}$$
$$= \frac{8\rho g^{3/2}}{v\sqrt{2\pi(x-x')}} \sigma \sigma' \theta \theta' \sin\left\{\frac{\pi}{4} - g(x-x')/v^2\right\} e^{-g(z+z')/v^2}.$$

Now if l is the length of the free wave which travels with the velocity of the ship

$$v^2 = gl/2\pi,$$

* 'Messenger of Mathematics,' 1896.

and the formula can be written

$$\frac{8g\rho\,\sigma\,\sigma'\,\theta\,\theta'}{\sqrt{l(x-x')}}\sin\pi\left\{\frac{1}{4}-2(x-x')/l\right\}e^{-2\pi(z+z')/l}.$$

This gives a maximum resistance, approximately, when

$$x-x'=(n+7/8)l \qquad [n \text{ an integer}]$$

and a maximum assistance when

$$x-x'=(n+3/8)l.$$

These formulæ correspond to the interference of the bow and stern waves, which has been so skilfully discussed by Mr. R. E. Froude. When the two elements are on the same vertical cross-section of the ship, another form of reduction may be given. Putting $x-x'=0$, the integral to be considered is

$$G=\int_1^\infty e^{-r\lambda^2}\frac{d\lambda}{\sqrt{\lambda^2-1}}.$$

Put $\lambda^2=\frac{1}{2}(1+\mu)$,

so that $d\lambda=\frac{1}{2\sqrt{2}}\frac{d\mu}{\sqrt{1+\mu}}$,

and $\lambda^2-1=\frac{1}{2}(\mu-1)$;

whence

$$G=\tfrac{1}{2}e^{-r/2}\int_1^\infty e^{-r\mu/2}\frac{d\mu}{\sqrt{\mu^2-1}};$$

or, if $\mu=\cosh\phi$,

$$G=\tfrac{1}{2}e^{-r/2}\int_0^\infty e^{-(r\cosh\phi)/2}\,d\phi$$
$$=\tfrac{1}{2}e^{-r/2}K_0(r/2),$$

where K_0 is the Bessel function, so indicated by Gray and Mathews (pp. 67, 90).

Hence

$$\int_1^\infty e^{-r\lambda^2}\frac{\lambda^2\,d\lambda}{\sqrt{\lambda^2-1}}=-\frac{dG}{dr}$$
$$=\tfrac{1}{4}e^{-r/2}\{K_0(r/2)-K_0'(r/2)\}$$
$$=\tfrac{1}{4}e^{-r/2}\{K_0(r/2)-K_1(r/2)\}.$$

since $K_0'=K_1$,

and the corresponding term in the resistance is

$$\frac{2\rho g^2 \sigma\sigma' \theta\theta'}{\pi v^2} e^{-g(z+z')/2v^2}\left[K_0\{g(z+z')/2v^2\} - K_1\{g(z+z')/2v^2\}\right].$$

The functions K do not appear to have been calculated, but their general course is known. It will be sufficient at present to refer to Gray and Mathews, chap. vii.

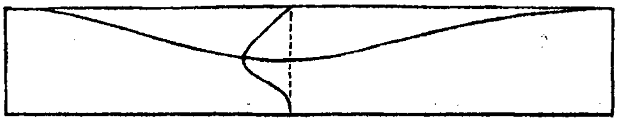
As an illustration of the process of calculating the resistance of a given ship at any speed, we may consider one of simple analytical form which is fairly ship-shape and easily realisable. Experimental confirmation of the result was not practicable, and the matter must be left in the hands of those who have the necessary apparatus at command.

Let the surface of the ship be

$$y = \pm c(1+\cos ax)(1+\cos bz),$$

between
$$x = \pm \pi/a,$$
$$z = 0 \quad \text{and} \quad \pi/b\,;$$

Fig. 2.



so that, for y positive,

$$f(x, z) = \frac{dy}{dx} = -ac \sin ax(1+\cos bz).$$

Here $I = 0$,

and
$$J = -ac\int_0^{\pi/b} (1+\cos bz)\, e^{-\lambda^2 z/k} dz \int_{-\pi/a}^{\pi/a} \sin ax \sin \lambda x/k\, dx$$

$$= -ac\frac{k}{\lambda^2}(2\lambda^4 + b^2k^2 - e^{-\pi\lambda^2/bk}\, b^2k^2)\frac{2ak^2}{(\lambda^4+b^2k^2)(a^2k^2-\lambda^2)}\sin \pi\lambda/ka,$$

where $k = v^2/g$; and therefore

$$R = \frac{16g\rho}{\pi} a^4c^2k^5 \int_1^\infty (2\lambda^4 + b^2k^2 - e^{-\pi\lambda^2/bk}\, b^2k^2)^2 \frac{\sin^2 \pi\lambda/ka}{(\lambda^4+b^2k^2)^2(a^2k^2-\lambda^2)^2}\frac{d\lambda}{\lambda^2\sqrt{\lambda^2-1}}$$

which is best calculated by mechanical quadrature.

Suppose, for example, in foot-second units

$v = 20$ (velocity of ship),
$2\pi/a = 200$ (length of ship),
$\pi/b = 20$ (depth below water-line),
$8c = 32$ (greatest breadth),

then the integral is found to be ·620, and the resistance is

$$R = 940 \text{ lbs. wt. about.}$$

This seems to be about what one would expect from the experimental results available; but I know of no formula with which to compare it, and experiment alone can decide whether the theory has numerical value. Of course the method of successive approximations can be applied if necessary.

To examine the case of the ship in shallow water in which all but the long waves are neglected, we may proceed as follows:—We make the motion steady as before and use the same set of axes. The pressure-equation gives at the surface

$$g\zeta + v\frac{d\phi}{dx} = 0, \text{ as before ;}$$

while the equation of continuity for long waves gives

$$\frac{d}{dx}\left\{(h-\zeta)\left(-v+\frac{d\phi}{dx}\right)\right\} + \frac{d}{dy}\left\{(h-\zeta)\frac{d\phi}{dy}\right\} = 0,$$

where h is the depth of the water, and this is

$$v\frac{d\zeta}{dx} + h\left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2}\right) = 0 \quad \text{(q. p.)}$$

Differentiating with respect to x and substituting for ϕ from the pressure-equation, we obtain

$$(gh - v^2)\frac{d^2\zeta}{dx^2} + gh\frac{d^2\zeta}{dy^2} = 0,$$

or

$$(c^2 - v^2)\frac{d^2\zeta}{dx^2} + c^2\frac{d^2\zeta}{dy^2} = 0,$$

where c is the velocity of free long waves.

The ship being wall-sided, and extending to the bottom of the water, the kinematic equation over the ship is

$$\frac{d\phi}{dy} = -v\frac{d\eta}{dx},$$

which on differentiation with respect to x and use of the pressure-equation gives

$$g\frac{d\zeta}{dy} = v^2\frac{d^2\eta}{dx^2},$$

and, of course,

$$\frac{d\zeta}{dy} = 0$$

over the rest of the plane $y = 0$.

Now if $v>c$, i.e. if the velocity of the ship is greater than that of the free wave, the equation

$$\frac{d^2\zeta}{dy^2}=\frac{v^2-c^2}{c^2}\frac{d^2\zeta}{dx^2}$$

is solved in the form

$$\zeta=\mathrm{F}\left(x+\sqrt{\frac{v^2-c^2}{c^2}}y\right), \quad . \quad . \quad . \quad . \quad (10)$$

where the boundary condition gives

$$\sqrt{\frac{v^2-c^2}{c^2}}\,\mathrm{F}'(x)=\frac{v^2}{g}\frac{d^2\eta}{dx^2},$$

or

$$\zeta=\mathrm{F}(x)=\frac{v^2c}{g\sqrt{v^2-c^2}}\frac{d\eta}{dx}. \quad . \quad . \quad . \quad . \quad (11)$$

The form of solution (10) is employed in order to make the diverging waves trail aft.

The disturbance therefore consists of two bands at an angle $\tan^{-1}(c/\sqrt{v^2-c^2})$ with the line of the ship's motion, the front of each band being a hump above the mean level and its back part a hollow, which is similar to the hump if the ship is similarly shaped fore and aft.

The resistance (R) is given by

$$\begin{aligned}\mathrm{R}&=-2h\int\delta p\,\frac{d\eta}{dx}\,dx\\&=2g\rho h\int\zeta\,\frac{d\eta}{dx}\,dx\\&=2\rho h\frac{v^2c}{\sqrt{v^2-c^2}}\int\left(\frac{d\eta}{dx}\right)^2dx,\end{aligned}$$

so that it is infinite when the velocity of the ship is equal to that of the free wave, and ultimately varies as the velocity.

If $v<c$, the differential equation for ζ takes the potential form

$$\frac{d^2\zeta}{dx^2}+\frac{d^2\zeta}{dy'^2}=0,$$

putting

$$y'=\frac{\sqrt{c^2-v^2}}{c}y.$$

The solution is now

$$\zeta(x',y')=\frac{1}{\pi}\frac{v^2}{g}\frac{c}{\sqrt{c^2-v^2}}\int\frac{d^2\eta}{dx^2}\log r'\,dx+\mathrm{C},$$

where $$r'^2 = (x'-x)^2+y'^2,$$

and there is no wave-resistance.

If we add the solutions for an equally-spaced infinite number of ships moving abreast, we get the case of a ship in the centre of a canal.

In the Table appended there may be an error of 1 in the last place or possibly of 2 in the values for $x>3$ or 4.

Melbourne University,
August 9, 1897.

TABLES of $\kappa J_0 - Y_0$ and $\kappa J_1 - Y_1$.

By Mr. B. A. SMITH, M.C.E.

x.	$\kappa J_0(x) - Y_0(x)$.	$\kappa J_1(x) - Y_1(x)$.	x.	$\kappa J_0(x) - Y_0(x)$.	$\kappa J_1(x) - Y_1(x)$.
·00	∞	∞	·41	·9243	2·7384
·01	4·7209	100·0261	·42	·8972	2·6822
·02	4·0274	50·0453	·43	·8706	2·6286
·03	3·6215	33·3951	·44	·8446	2·5773
·04	3·3331	25·0767	·45	·8190	2·5282
·05	3·1091	20·0903	·46	·7940	2·4813
·06	2·9258	16·7695	·47	·7694	2·4362
·07	2·7705	14·4002	·48	·7453	2·3929
·08	2·6359	12 6255	·49	·7216	2·3514
·09	2·5168	11·2470	·50	·6983	2·3114
·10	2·4099	10 1457	·51	·6753	2 2729
·11	2·3133	9·2459	·52	·6528	2·2357
·12	2·2245	8·4971	·53	·6306	2·1999
·13	2·1428	7·8645	·54	·6088	2·1653
·14	2·0670	7·3230	·55	·5873	2·1319
·15	1·9961	6·8545	·56	·5661	2·0995
·16	1·9297	6·4450	·57	·5453	2·0681
·17	1·8671	6·0843	·58	·5248	2·0377
·18	1 8079	5·7642	·59	·5046	2·0083
·19	1·7517	5·4780	·60	·4846	1·9798
·20	1·6982	5·2209	·61	·4650	1·9521
·21	1·6472	4·9888	·62	·4456	1·9251
·22	1·5983	4·7779	·63	·4264	1·8988
·23	1·5515	4·5855	·64	·4076	1·8732
·24	1·5066	4·4094	·65	·3890	1·8483
·25	1·4633	4·2476	·66	·3707	1·8241
·26	1·4216	4 0983	·67	·3525	1·8005
·27	1·3813	3 9603	·68	·3346	1·7775
·28	1·3424	3·8323	·69	·3169	1·7550
·29	1·3046	3·7131	·70	·2995	1·7329
·30	1·2680	3·6020	·71	·2823	1·7114
·31	1·2326	3·4982	·72	·2653	1·6904
·32	1·1981	3·4007	·73	·2485	1·6699
·33	1·1645	3·3094	·74	·2319	1·6496
·34	1·1319	3·2233	·75	·2155	1·6299
·35	1·1000	3·1423	·76	·1993	1·6105
·36	1·0690	3·0656	·77	·1833	1·5914
·37	1·0387	2·9932	·78	·1675	1·5728
·38	1·0092	2·9245	·79	·1518	1·5544
·39	·9803	2·8593	·80	·1363	1·5365
·40	·9519	2·7973	·81	·1211	1·5188